An optimized single-step method for integrating Cauchy problems

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Abstract. This work involves development of an optimum third order single-step explicit method for Cauchy problems. The proposed method is analyzed for consistency, stability, local and global error bounds, and convergence. Further, numerical investigation is carried out to assess effectiveness of the method in comparison to existing numerical schemes, including Modified Improved Modified Euler (MIME) method, Third order Euler method (TOEM) and classical Runge-Kutta method of order three (RK3). The testing factors are error and CPU time which have been computed using Matlab R2014b. It is observed that the proposed method possesses minimum error bounds; and is also favourable in terms of both accuracy and computational cost.

AMS (MOS) Subject Classification Codes: 65-02; 65L-05; 65L-12; 65L-20; 65L-70

Key Words: Consistency, Stability, Convergence, Error bounds.

1. INTRODUCTION

Mathematical modeling of many real world phenomena often leads to ordinary or partial differential equations. Most of these equations are highly nonlinear and the exact solutions are not always possible. Therefore, researchers in engineering, applied mathematics and other scientific fields often resort to numerical techniques which provide approximate solutions. Numerical methods for the solution of nonlinear Cauchy problems are efficient tools that have been gradually developed by many mathematicians since the eighteenth century. Later on, with the invention of high speed digital computers in the late twentieth century, the study of various phenomena in engineering and science has been made much easier and efficient by the use of numerical techniques. These techniques have proved their efficacy in a number of physical problems, for example, chemical kinetics [8, 14, 15],
fluid dynamics [11, 25, 30, 32], electrical circuits [5, 13, 27], computer virology [3, 21], image processing [26] and epidemiology [4, 10, 17, 19, 28, 29]. There has been extensive study of single-step explicit methods for the numerical integration of Cauchy problems [1, 2, 7, 12, 18, 20, 22, 23]. This further stimulates the interest in their theoretical and numerical investigation while carrying out computational studies for the real world application problems. In this work, an explicit single-step method of third order is developed for initial value problems of ordinary differential equations also known as Cauchy problems. Although, there exist higher order methods but with an increased number of slope evaluations per step. For example, the fourth order RK method requires four and the fifth order method involves six slope evaluations; thereby increasing the time for numerical simulation. This is the major concern in the present study - to develop a third order method which offers significant accuracy while consuming less CPU time. The proposed method is analyzed for consistency, stability and convergence. We investigate performance of the method on the basis of stability interval, local and global error bounds, accuracy and computational time in comparison to the following methods:

- **Modified Improved Modified Euler (MIME) method** [20]: It is a second order accurate method involving three function evaluations per integration step. The iterative scheme for MIME method is given as:
  \[
  u_r = u_{r-1} + d_3 \\
  \begin{align*}
  d_1 &= \Delta t g (t_{r-1}, u_{r-1}) \\
  d_2 &= \Delta t g \left( t_{r-1}, u_{r-1} + \frac{d_1}{2} \right) \\
  d_3 &= \Delta t g \left( t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_2}{2} \right)
  \end{align*}
  \]  
  (1.1)

- **Third Order Euler Method (TOEM)** [1]: It involves three function evaluations per step and is given by the following iterative scheme:
  \[
  u_r = u_{r-1} + d_3 \\
  \begin{align*}
  d_1 &= \Delta t g (t_{r-1}, u_{r-1}) \\
  d_2 &= \Delta t g \left( t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_1}{2} \right) \\
  d_3 &= \Delta t g \left( t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_2}{2} \right)
  \end{align*}
  \]  
  (1.2)

- **Classical Runge-Kutta method of third order (RK3)** [24]: Proposed by Runge in 1895, it involves three function evaluations per integration step as given in ( 1.3):
  \[
  u_r = u_{r-1} + \frac{1}{6} (d_1 + 4d_2 + d_3) \\
  \begin{align*}
  d_1 &= \Delta t g (t_{r-1}, u_{r-1}) \\
  d_2 &= \Delta t g \left( t_{r-1} + \frac{\Delta t}{2}, u_{r-1} + \frac{d_1}{2} \right) \\
  d_3 &= \Delta t g (t_{r-1} + \Delta t, u_{r-1} - d_1 + 2d_2)
  \end{align*}
  \]  
  (1.3)
It is found that TOEM, RK3 and proposed method all have the same interval of stability which is larger than that of MIME. The proposed method possesses the minimum error bounds followed by RK3, TOEM and MIME, and also outperforms these methods with respect to accuracy and CPU time.

2. PROPOSED METHOD

Let

\[ u_r = u_{r-1} + c_1 d_1 + c_2 d_2 + c_3 d_3 \]
\[ d_1 = \Delta t g(t_{r-1}, u_{r-1}) = \Delta t g \]
\[ d_2 = \Delta t g(t_{r-1} + a_2 \Delta t, u_{r-1} + b_{21} d_1) \]
\[ d_3 = \Delta t g(t_{r-1} + a_3 \Delta t, u_{r-1} + b_{31} d_1 + b_{32} d_2) \]

Here \( a_2, a_3, b_{21}, b_{31}, b_{32}, c_1, c_2 \) and \( c_3 \) are unknowns. Taylor series expansion of (2.4) yields

\[ u_r = u_{r-1} + \left( c_1 + c_2 + c_3 \right) \Delta t g + \left( a_2 c_2 + a_3 c_3 \right) (\Delta t)^2 g_t + \left\{ b_{21} c_2 \right. \]
\[ + \left. (b_{31} + b_{32}) c_3 \right\} (\Delta t)^2 g_{uu} + \frac{1}{2} \left( a_2^2 c_2 + a_3^2 c_3 \right) (\Delta t)^3 g_{tt} + \left\{ b_{21}^2 c_2 \right. \]
\[ + \left. (b_{31} + b_{32})^2 c_3 \right\} (\Delta t)^3 g_{uuu} + \frac{1}{2} \left( a_2 b_{21} c_2 \right. \]
\[ + \left. (a_3 b_{31} + a_3 b_{32}) c_3 \right\} (\Delta t)^3 g_{tu} + \frac{1}{2} \left( b_{21}^2 c_2 \right. \]
\[ + \left. (b_{31} + b_{32})^2 c_3 \right\} (\Delta t)^3 g_{uu} + \left. b_{21} b_{32} c_3 (\Delta t)^3 g_{uuu} \right. \]
\[ + \left. O(\Delta t)^4 \right. \]

The Taylor’s expansion of \( u_r \) is given by

\[ u_r = u_{r-1} + \Delta t g + \frac{(\Delta t)^2}{2} (g_t + g_{uu}) + \frac{(\Delta t)^3}{6} (g_{tt} + 2 g_{tu} + g_{uu} + g_{tt} g_{uu}) \]
\[ + \left. O(\Delta t)^4 \right. \]

Comparing the coefficients of \( \Delta t \) in (2.5) as far as \( (\Delta t)^3 \) with the Taylor’s expansion (2.6) yields the following order conditions:

\[ c_1 + c_2 + c_3 = 1 \]
\[ a_2 c_2 + a_3 c_3 = \frac{1}{2} \]
\[ b_{21} c_2 + (b_{31} + b_{32}) c_3 = \frac{1}{2} \]
\[ a_2^2 c_2 + a_3^2 c_3 = \frac{1}{3} \]
\[ a_2 b_{21} c_2 + (a_3 b_{31} + a_3 b_{32}) c_3 = \frac{1}{3} \]
\[ b_{21}^2 c_2 + (b_{31} + b_{32})^2 c_3 = \frac{1}{3} \]
\[ a_2 b_{32} c_3 = \frac{1}{6} \]
\[ b_{21} b_{32} c_3 = \frac{1}{6} \]
An optimal solution of (2.7) leads to following third order accurate method:

\[ u_0 = u_{r-1} + \frac{1}{4} (d_1 + 3d_3) \]
\[ d_1 = \Delta t \psi(t_{r-1}, u_{r-1}) \]
\[ d_2 = \Delta t \psi(t_{r-1} + \frac{2}{3} \Delta t, u_{r-1} + \frac{2}{3} d_1) \]
\[ d_3 = \Delta t \psi(t_{r-1} + \frac{1}{3} \Delta t, u_{r-1} + \frac{1}{3} (d_1 + d_2)) \] (2.8)

Here, by optimal solution we mean the values of the unknowns present in the nonlinear system (2.7) which give rise to a numerical scheme with minimum error bounds and maximum accuracy in comparison to other methods under consideration. The subsequent sections have been accordingly expanded to delineate this notion of optimal solution, and hence an optimal method.

3. ANALYSIS OF PROPOSED METHOD

3.1. Consistency.

**Definition 1.** A single-step method has the form \( u_0 = u_{r-1} + \Delta t \psi(t_{r-1}, u_{r-1}; \Delta t) \) where \( \psi(t_{r-1}, u_{r-1}; \Delta t) \) is known as the increment function of the method.

**Definition 2.** Given an initial value problem \( \frac{du}{dt} = g(t, u) \); \( u(t_0) = u_0 \), a numerical method with an increment function \( \psi(t, u; \Delta t) \) is said to be consistent if \( \lim_{\Delta t \to 0} \psi(t, u; \Delta t) = g(t, u) \).

Applying these definitions to the increment function \( \psi \) of proposed method,

\[ \lim_{\Delta t \to 0} \psi(t_{r-1}, u_{r-1}; \Delta t) = \lim_{\Delta t \to 0} \frac{\Delta t}{\Delta t} (d_1 + 3d_3) = g(t_{r-1}, u_{r-1}) \]

Hence, the method is consistent.

Next, we state and prove two important theorems which are used to demonstrate stability and convergence of proposed method.

**Theorem 1.** Let \( \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n \) be real numbers satisfying \( |\gamma_{i+1}| \leq (1 + \alpha)|\gamma_i| + \beta \), with \( \alpha > 0, \beta \geq 0, i = 0, 1, 2, \ldots, n-1 \), then \( |\gamma_n| \leq e^{n\alpha}|\gamma_0| + \frac{e^{n\alpha}-1}{\alpha}\beta \).

**Proof.** From the assumptions, we get

\[ |\gamma_1| \leq (1 + \alpha)|\gamma_0| + \beta \]
\[ |\gamma_2| \leq (1 + \alpha)|\gamma_1| + \beta = (1 + \alpha)^2 |\gamma_0| + (1 + \alpha)\beta + \beta \]
\[ \vdots \]
\[ |\gamma_n| \leq (1 + \alpha)^n |\gamma_0| + \beta |1 + (1 + \alpha) + (1 + \alpha)^2 + \cdots + (1 + \alpha)^{n-1}| \]
\[ \leq e^{n\alpha}|\gamma_0| + \beta \frac{e^{n\alpha}-1}{\alpha} \cdot 0 < 1 + \alpha \leq e^{\alpha} \text{ for } \alpha > -1. \] □

**Theorem 2.** Suppose \( (t_{r-1}, u_{r-1}) \) and \( (t_{r-1}, \tilde{u}_{r-1}) \) are any two points in the region \( T \) defined by \( T = \{ (t, u) \in \mathbb{R}^2 | t_0 \leq t \leq t_n, -\infty < u < \infty \} \), and \( g \) is a Lipschitz continuous function on \( T \) such that \( |g(t_{r-1}, u_{r-1}) - g(t_{r-1}, \tilde{u}_{r-1})| \leq L |u_{r-1} - \tilde{u}_{r-1}| \), then the increment function \( \psi \) is Lipschitz continuous, and \( |\psi(t_{r-1}, u_{r-1}; \Delta t) - \psi(t_{r-1}, \tilde{u}_{r-1}; \Delta t)| \leq L |u_{r-1} - \tilde{u}_{r-1}| \) where \( L \) and \( \tilde{L} \) are, respectively, the Lipschitz constants for \( g \) and \( \psi \).
From the assumptions, we have:

\begin{align*}
|d_1 - \hat{d}_1| &= \Delta t \left| g(t_{r-1}, u_{r-1}) - g(t_{r-1}, \hat{u}_{r-1}) \right| \
|d_2 - \hat{d}_2| &= \Delta t \left| g(t_{r-1} + \frac{2}{3} \Delta t, u_{r-1} + \frac{2}{3} \hat{d}_1) - g(t_{r-1} + \frac{2}{3} \Delta t, \hat{u}_{r-1} + \frac{2}{3} \hat{d}_1) \right| \
&\leq L\Delta t \left| (u_{r-1} - \hat{u}_{r-1}) \right| + \frac{2}{3} L\Delta t \left| d_1 - \hat{d}_1 \right| \
&\leq \left( L\Delta t + \frac{2}{3} L^2 (\Delta t)^2 \right) |u_{r-1} - \hat{u}_{r-1}|
\end{align*}

Hence, the increment function \( \psi \) of the proposed method:

\[
\psi(t; u; \Delta t) = \frac{1}{\Delta t} \left( d_1 + 3d_3 \right) - \left( \frac{1}{\Delta t} d_1 + 3d_3 \right)
\]

\[
\leq \frac{1}{\Delta t} \left| d_1 - \hat{d}_1 \right| + \frac{3}{\Delta t} \left| d_3 - \hat{d}_3 \right|
\]

\[
\leq \bar{L} |u_{r-1} - \hat{u}_{r-1}|; \quad \bar{L} = L + \frac{2}{3} L^2 \Delta t + \frac{1}{3} L^3 (\Delta t)^2
\]

This implies that the increment function of proposed method is Lipschitz continuous. \( \square \)

3.2. Stability.

**Theorem 3.** Let \( u_r \) and \( \hat{u}_r \) be two solutions to the differential equation \( u' = g(t, u) \), generated by a numerical method, subject to the initial conditions \( u(t_0) = u_0 \) and \( \hat{u}(t_0) = \hat{u}_0 \) respectively, such that \( |u_0 - \hat{u}_0| < \varepsilon, \varepsilon > 0 \). The condition \( |u_r - \hat{u}_r| \leq K \left| u_0 - \hat{u}_0 \right|, K > 0 \) is the necessary and sufficient condition for the method to be stable.

Applying Theorem 2 to the proposed method gives,

\[
\begin{align*}
|u_r - \hat{u}_r| &= |u_{r-1} + \Delta t \psi(t_{r-1}, u_{r-1}; \Delta t) - \hat{u}_{r-1} - \Delta t \psi(t_{r-1}, \hat{u}_{r-1}; \Delta t)| \\
&\leq \left( 1 + \bar{L}\Delta t \right) |u_{r-1} - \hat{u}_{r-1}| \\
&= \left( 1 + \bar{L}\Delta t \right) |u_0 - \hat{u}_0|
\end{align*}
\]

Using Theorem 1 with \( \alpha = \bar{L}\Delta t \) and \( \beta = 0 \) gives

\[
|u_r - \hat{u}_r| \leq K \left| u_0 - \hat{u}_0 \right|, K = e^{\bar{L}\Delta t}
\]

This establishes that the proposed method is stable.

Applying proposed method to Dahlquist’s model problem [9]

\[
u' = \lambda u, \lambda \in \mathbb{C}, R_e(\lambda) < 0
\]

(3.9)

gives

\[
u_r = \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} \right) u_{r-1}
\]

(3.10)

where \( z = \lambda \Delta t \). The ratio \( u_r / u_{r-1} \) is called stability function \( \phi(z) \). Hence, stability function of the proposed method is...
$$\phi(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} \quad (3.11)$$

It is found that $|\phi(z)| \leq 1$ for $-2.51 \leq R_e(z) \leq 0$. Table 1 compares stability function and interval of stability of proposed method with those of MIME, TOEM and RK3 methods. While TOEM and proposed method each have same interval of stability as that of conventional RK3 method, MIME possesses a smaller stability interval.

<table>
<thead>
<tr>
<th>Method</th>
<th>Linear stability function</th>
<th>Interval of linear stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>MIME</td>
<td>$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$</td>
<td>(-2, 0)</td>
</tr>
<tr>
<td>TOEM</td>
<td>$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$</td>
<td>(-2.51, 0)</td>
</tr>
<tr>
<td>RK3</td>
<td>$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$</td>
<td>(-2.51, 0)</td>
</tr>
<tr>
<td>Proposed</td>
<td>$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$</td>
<td>(-2.51, 0)</td>
</tr>
</tbody>
</table>

3.3. Local Error Bounds. The local truncation error for proposed method is

$$\tau_{r+1} = \frac{(\Delta t)^4}{216} \left( g_{ttt} + 3g_{ttu} + 3g^2_{tuu} + g^3_{uuu} + 3g_{gtu} + 3g_{gtu} - 3g_{u} g_{tu} - 3g_t g_u + 9g_t g^2_u + 9g^3_u \right) + O(\Delta t)^5 \quad (3.12)$$

Thus, the principal error function is given as

$$\Phi(t_r, u_r) = \frac{1}{216} \left( g_{ttt} + 3g_{ttu} + 3g^2_{tuu} + g^3_{uuu} + 3g_{gtu} + 3g_{gtu} - 3g_{u} g_{tu} - 3g_t g_u + 9g_t g^2_u + 9g^3_u \right) \quad (3.13)$$

Lotkin [16] proposed following bounds for the function $g$ and its partial derivatives for $t \in [a, b]$ and $u \in (-\infty, \infty)$:

$$|g(t, u)| < M, \quad \left| \frac{\partial^{i+j}g}{\partial t^i \partial u^j} \right| < \frac{N^{i+j}}{M^{j-1}}, \quad i + j \leq p$$

where $M$ and $N$ are positive constants and $p$ is order of accuracy. Using Lotkin’s bounds,

$$|\Phi(t_r, u_r)| < \frac{19}{108} M N^3 (\Delta t)^4$$

Hence, the bound on local truncation error $\tau_{r+1}$ of the proposed method is obtained as

$$|\tau_{r+1}| < T, \quad T = \frac{19}{108} M N^3 (\Delta t)^4$$

3.4. Convergence. Consider the proposed method:

$$u_{i+1} - u_i - \Delta t \psi(t_i, u_i; \Delta t) = 0 \quad (3.14)$$

The quantity

$$\tau_i = u(t_{i+1}) - u(t_i) - \Delta t \psi(t_i, u(t_i); \Delta t) \quad (3.15)$$
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is local error of the proposed method. On subtracting (3.14) from (3.15), it is deduced that

\[ |e_{i+1}| \leq |e_i| + \Delta t \left| \psi(t_i, u(t_i); \Delta t) - \psi(t_i, u_i; \Delta t) \right| + |\tau_i| \]  

(3.16)

where \( e_i \) is global error. Using Theorems 1 and 2, (3.16) yields

\[ |e_{i+1}| \leq \left(1 + \Delta t \hat{L}\right) |e_i| + |\tau_i| \]  

(3.17)

Let \( T = \max |\tau_i|, i = 0, 1, 2, \ldots, k - 1 \), then Theorem 1 implies that

\[ |e_r| \leq e^{\hat{L}T} |e_0| + \frac{T}{\Delta t} \left[e^{\hat{L}T} - 1\right] \]  

(3.18)

Finally, the global error bound for the proposed method is obtained as

\[ |e_r| \leq \frac{T}{\Delta t} \left[e^{\hat{L}(t_r-t_0)} - 1\right] \]  

(3.19)

where \( T \) is its local error bound. As \( \lim_{\Delta t \to 0} |e_r| = 0 \), this implies convergence of our method. Table 2 provides comparison of local and global error bounds of proposed method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Local Error Bound</th>
<th>Global Error Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>MIME</td>
<td>( \frac{1}{3} M N^2 (\Delta t)^3 ) ( e^{\hat{L}_{\text{MIME}}(t_r-t_0)} ) ( -1 )</td>
<td>( \frac{1}{3} M N^2 (\Delta t)^3 ) ( e^{\hat{L}_{\text{MIME}}(t_r-t_0)} ) ( -1 )</td>
</tr>
<tr>
<td>TOEM</td>
<td>( \frac{53}{144} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{TOEM}}(t_r-t_0)} ) ( -1 )</td>
<td>( \frac{53}{144} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{TOEM}}(t_r-t_0)} ) ( -1 )</td>
</tr>
<tr>
<td>RK3</td>
<td>( \frac{1}{4} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{RK3}}(t_r-t_0)} ) ( -1 )</td>
<td>( \frac{1}{4} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{RK3}}(t_r-t_0)} ) ( -1 )</td>
</tr>
<tr>
<td>Proposed</td>
<td>( \frac{19}{108} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{Proposed}}(t_r-t_0)} ) ( -1 )</td>
<td>( \frac{19}{108} M N^3 (\Delta t)^4 ) ( e^{\hat{L}_{\text{Proposed}}(t_r-t_0)} ) ( -1 )</td>
</tr>
</tbody>
</table>

\( \hat{L}_{\text{MIME}}, \hat{L}_{\text{TOEM}}, \hat{L}_{\text{RK3}} \) and \( \hat{L}_{\text{Proposed}} \) respectively denote the Lipschitz constant for the increment functions of MIME, TOEM, RK3 and proposed methods.

\[ \hat{L}_{\text{MIME}} = L + \frac{1}{2} L^2 \Delta t + \frac{1}{2} L^2 (\Delta t)^2, \]

\[ \hat{L}_{\text{TOEM}} = \hat{L}_{\text{RK3}} = \hat{L}_{\text{Proposed}} = L + \frac{1}{2} L^2 \Delta t + \frac{1}{6} L^3 (\Delta t)^2 \]

with those of MIME, TOEM and RK3 methods. A simple analysis of these error bounds leads to the following observation:

For given values of \( M, N \) and \( \Delta t \);  
\( \text{Error Bound}_{\text{Proposed}} < \text{Error Bound}_{\text{RK3}} < \text{Error Bound}_{\text{TOEM}} < \text{Error Bound}_{\text{MIME}} \)

which holds for both local and global error bounds.

4. Numerical Examples

4.1. Example 1. \( \frac{du}{dt} = u - tu^2 \) subject to \( u(0) = 1 \). The exact solution is \( u(t) = \frac{1}{2e^{-t} + t - 1} \).
For the nonlinear problem under consideration, we present a comparison of absolute errors yielded by MIME, TOEM, RK3 and proposed method at \( t = 1 \) corresponding to various step sizes in Table 3, whereas, Table 4 presents the CPU time consumed by these methods against different values of error tolerance at the same point of domain. It can be clearly seen that the proposed method not only produces the smallest errors for each step size value, but also takes the least computational time to attain the desired accuracy.

4.2. Example 2. \[
\frac{du_1}{dt} = u_2^2 - 2u_1, \quad \frac{du_2}{dt} = u_1 - u_2 - tu_2^2
\]
subject to \( u_1(0) = 0 \) and \( u_2(0) = 1 \). The exact solution is \( u_1(t) = te^{-2t}, u_2(t) = e^{-t} \).

Table 5 shows \( L_2 \) error norm values given by MIME, TOEM, RK3 and proposed method at \( t = 2 \) for the nonlinear system under consideration. It can be seen that the minimum error norm values are obtained from the proposed method followed by RK3, TOEM and MIME methods. In terms of CPU time, proposed method is found to be efficient in contrast to both MIME and TOEM methods and is comparable to RK3 as illustrated from Table 6.
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4.3. Example 3 - Application in population dynamics.

\[
\frac{du}{dt} = a\left(1 - \frac{u}{C}\right)u; \quad u(0) = u_0
\]  

(4.20)

The nonlinear autonomous Cauchy problem (4.20) is the population growth model known as Verhulst or logistic equation [31]. The positive constants \(a\) and \(C\) are respectively known as intrinsic growth rate and saturation level for the given population. The analytical solution is given as:

\[
u(t) = \frac{u_0 C}{u_0 + (C - u_0)e^{-at}}
\]  

(4.21)

In [6, p. 80], logistic equation is applied on Pacific halibut growth where \(u\) denotes the biomass (in kilogram) of halibut fishery and \(t\) is time. The parameter values used are \(a = 0.71/\text{year}\) and \(C = 8.05 \times 10^7 \text{ kg}\) with initial biomass \(u_0 = 2.0125 \times 10^7 \text{ kg}\). We integrate logistic equation from \(t = 0\) to \(t = 2\) subject to these conditions using MIME, TOEM, RK3 and proposed method. The results are presented in Tables 7 and 8.

<table>
<thead>
<tr>
<th>Step size</th>
<th>MIME</th>
<th>TOEM</th>
<th>RK3</th>
<th>Proposed Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>3.55E-01</td>
<td>2.72E-01</td>
<td>4.22E-05</td>
<td>9.34E-06</td>
</tr>
<tr>
<td>0.01</td>
<td>3.55E+01</td>
<td>2.72E-01</td>
<td>4.39E-02</td>
<td>1.08E-02</td>
</tr>
<tr>
<td>0.05</td>
<td>8.84E+02</td>
<td>6.77E+02</td>
<td>5.56E+00</td>
<td>1.30E+00</td>
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<table>
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<th>Error tolerance</th>
<th>MIME</th>
<th>TOEM</th>
<th>RK3</th>
<th>Proposed Method</th>
</tr>
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<td>1.74E+00</td>
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From the tables, it is evident that the performance of proposed method is far better than each of MIME, TOEM and RK3 when implemented to a physical problem. The proposed method leads to the minimal absolute errors, at the final time $t = 2$, corresponding to different step-size values. It also shows the fastest convergence by consuming the least amount of CPU time to achieve the specified accuracy levels. It is particularly noticeable here that both MIME and TOEM give poor accuracy as depicted from the high error values at even small step size of 0.01. Moreover, both these methods also prove to be computationally expensive.

5. Conclusion

In this work, an optimal third order accurate scheme has been developed for solving Cauchy problems. First, an abstract analysis of the proposed scheme is carried out for stability, consistency, convergence and error bounds. The proposed method is then implemented on three nonlinear problems for analyzing its performance in terms of accuracy and computational time in contrast to three existing methods – MIME, TOEM and Classical RK3. The observations drawn are as follows:

(i) TOEM, RK3 and proposed methods all have the same interval of stability which is larger than that of MIME.
(ii) Proposed method has the minimum local and global error bounds followed by RK3, TOEM and MIME methods.
(iii) In terms of accuracy, proposed method yields errors significantly smaller in magnitude than resulting from each of RK3, TOEM and MIME.
(iv) Overall, the proposed scheme proves to be computationally cost effective as it consumes considerably less CPU time while attaining the desired level of accuracy.

6. Authors’ Contribution

The idea of this work is given by first author. All authors have contributed equally towards preparation of this article.

7. Acknowledgement

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8. Declaration of Interest

‘Declarations of interest: none’

9. Role of the Funding Source

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