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# Geometry of Tangential and Configuration Chain Complexes for Higher Weights

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**Abstract.** In this study, the geometry of a first order tangent group and of configuration chain complexes is proposed. First, the morphisms are introduced to define the geometry for weight $n = 4$, and then this geometry is extended for higher weight $n = 5$. All associated commutative diagrams are presented.

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## 1. Introduction

Researchers have often used configuration spaces and its chain complex as a tool in active areas of pure mathematics. In the Grassmannian configuration chain complex, for example, free abelian groups are connected through two types of differential boundary maps $d$ and $p$ [22]. Configuration spaces are naturally related to polylogarithmic groups and its chain complexes; that is why many researchers have tried to find the relationship between configurations and polylogarithmic group chain complexes.
Bloch [1] defined polylogarithmic group $\mathcal{B}(F)$ for weight 1; it is a quotient of Z-module $Z[F^{\cdot\cdot}]$ and Abel's five terms relation. For weight 2, Bloch [1] defined a group denoted by $\mathcal{B}_2(F)$, generated by the cross ratio of four points, and introduced a chain complex called Bloch-Suslin complex.

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times$$

Goncharov introduced the concept of triple cross ratio to define the group $\mathcal{B}_3(F)$ for weight 3. He further generalized the Bloch group as $\mathcal{B}_n(F)$ creating a generalized version of the

Bloch-Suslin complex that was called the Goncharov Complex

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_2} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta_1} \wedge^n(F^\times)$$

First, Goncharov found geometry between configuration and Bloch-Suslin polylogarithmic chain complex through homomorphisms for weight 2 and proved the commutativity of the associated diagram [7]. Goncharov also extended his work to define the geometry of configuration and the generalized polylogarithmic chain complex called Goncharov complex for weight 3 [7]. On the other hand, Khalid et al. [12–15, 17–19] introduced the generalized geometry between Goncharov polylogarithmic complex and configuration chain complex for any weight n.
Later, Cathelineau introduced variant of Goncharov complex called Cathelineau complex in two different ways: one was infinitesimal while other was in a tangential setting [2, 3]. $\beta_n(F)$ was the group for infinitesimal chain complex and $T\mathcal{B}_n(F)$ for tangential complex. Cathelineau first introduced the Tangent to Bloch-Suslin complex $T\mathcal{B}_2(F) \to F \otimes F^\times \oplus \wedge^2 F^\times$ and then generalized the former.

$$T\mathcal{B}_n(F) \xrightarrow{\delta_{n,\varepsilon}} \cdots \xrightarrow{\delta_{1,\varepsilon}} \begin{array}{c} T\mathcal{B}_2(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times \end{array} \xrightarrow{\delta_\varepsilon} (F \otimes \wedge^{n-1}F^\times) \oplus (\wedge^n F)$$

Siddiqui introduced both the cross ratio of four points and the famous Siegel's cross-ratio properties in tangential form and also showed that the Goncharovs projected five term relation can also be defined for tangent group $T\mathcal{B}_2(F)$ [20]. With the help these constructions, Siddiqui [20] defined the morphisms to connect the configuration sub complex and first order tangential chain complex for both weight 2 and 3, in order to come up with commutative diagrams [20].
Hussain [11] introduced second and third order tangent groups denoted by $T\mathcal{B}_2^2(F)$ and $T\mathcal{B}_3^2(F)$ for weight 2 and 3. Hussain [11] also found the relation of these groups with configuration chain complexes through morphisms and proved the commutativity of the associated diagrams.
Here in this article, some interesting morphisms are introduced to define the new geometry of configuration and tangential chain complexes for higher weights 4 and 5. Section 2 describes the basic concepts of configuration chain complexes, truncated polynomial ring, cross ratio in dual numbers, classical polylogarithmic groups complexes, first order tangent group and generalized tangential groups chain complex, geometry between configuration, and the tangential complexes for weight 2 and 3. Section 3 provides the geometry and commutative diagrams of the configuration and tangential complexes for weight 4 and 5. The last section concludes the entire research work.

## 2. Preliminaries and Basic Concepts

2.1. **Grassmannian Configuration Chain Complex.** Let us have $GL_n(F)$ be a general linear group of order $n$, acting diagonally on a set $V^n$. The elements of group action $GL_n(F) * V^n = V^n$ are $(v_0, ..., v_n)$ called configurations of $n$ vectors in $n$-dimensional vector space $V$ defined some arbitrary field F.
Consider a free abelian group $G_n(V)$ generated by all possible projective configuration of

$n$ points $(v_1, ..., v_n) \in V^n$. Let $d$ be a differential boundary morphism, defined as

$$d : (v_0, ..., v_n) = \sum_{i=0}^{n} (-1)^i (v_0, ..., \hat{v}_i, ..., v_n) \tag{2.1}$$

Another differential map $p$ is defined as

$$p : (v_0, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i (v_i | v_0, \ldots, \hat{v}_i, \ldots, v_n) \tag{2.2}$$

Suslin [22] connected above free abelian groups using above two differential morphisms in following way to define Grassmannian configuration chain complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_m(n) & \xrightarrow{d} & G_{m-1}(n) & \xrightarrow{d} & G_{m-2}(n) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{m-1}(n-1) & \xrightarrow{d} & G_{m-2}(n-1) & \xrightarrow{d} & G_{m-3}(n-1) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots & \xrightarrow{d} & G_{m-2}(n-2) & \xrightarrow{d} & G_{m-3}(n-2) & \xrightarrow{d} & G_{m-4}(n-2)
\end{array} \tag{A}$$

The above diagram is bi-complex and each square is commutative (see [16, 22]).

2.2. **Tangential Configuration Spaces.** Assume that F be a field with 0 characteristic, we define the ring of $k^{th}$ truncated polynomial by $F[\varepsilon]_k := F[\varepsilon]/\varepsilon^k, k \geq 1$. Now define an affine space $\mathbb{A}^n_{F[\varepsilon]_n}$ defined over truncated polynomial $F[\varepsilon]_n$. Assume $v = (a_1, a_2, a_3, ..., a_n)^t \in \mathbb{A}^n_F \backslash (0, 0, 0, ..., 0)^t$ and $v_\varepsilon = (a_{1,\varepsilon}, a_{2,\varepsilon}, a_{3,\varepsilon}, ..., a_{n,\varepsilon})^t \in \mathbb{A}^n_F$ also $v_{\varepsilon^n} = (a_{1,\varepsilon^{k-1}}, a_{2,\varepsilon^{k-1}}, a_{3,\varepsilon^{k-1}}, ..., a_{n,\varepsilon^{k-1}})^t \in \mathbb{A}^n_F$ [11].
Let $G_m(\mathbb{A}^n_{F[\varepsilon]_k})$ be a free abelian group generated by $(v_1^*, v_2^*, v_3^*, ..., v_m^*)$ m vectors in affine space $(\mathbb{A}^n_{F[\varepsilon]_k})$ [11], where the element $v^* = v + v_\varepsilon \varepsilon + ... + v_{\varepsilon^{k-1}} \varepsilon^{k-1}$. Now define the a boundary map

$$d : G_m(\mathbb{A}^n_{F[\varepsilon]_k}) \to G_{m-1}(\mathbb{A}^n_{F[\varepsilon]_k})$$

and another differential map

$$p : G_m(\mathbb{A}^n_{F[\varepsilon]_k}) \to G_{m-1}(\mathbb{A}^{n-1}_{F[\varepsilon]_k})$$

with the help of these maps following is Grassmannian tangential configuration chain complex

$$\begin{array}{ccccc}
G_m(\mathbb{A}^n_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-1}(\mathbb{A}^n_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-2}(\mathbb{A}^n_{F[\varepsilon]_k}) \\
\downarrow p & & \downarrow p & & \downarrow p \\
G_{m-1}(\mathbb{A}^{n-1}_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-2}(\mathbb{A}^{n-1}_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-3}(\mathbb{A}^{n-1}_{F[\varepsilon]_k})
\end{array} \tag{B}$$

2.3. **Cross Ratio.** Let us define the cross ratio of $4$ points as

$$r(v_0, v_1, v_2, v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}$$

where $(v_0, v_1, v_2, v_3) \in A_F^2$ or $\mathbf{P}_F^1$. Siegel [21] defined the following most important property of ratio

$$1 = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)} + \frac{\triangle(v_0, v_1)\triangle(v_2, v_3)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}. \tag{2.3}$$

Or

$$\frac{\triangle(v_0, v_2)\triangle(v_1, v_3) - \triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)} = \frac{\triangle(v_0, v_1)\triangle(v_2, v_3)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}. \tag{2.4}$$

2.3.1. *Cross Ratio in $F[\varepsilon]_k$.* First we consider the following cases

**a :** For n=2 and k=1,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} = \triangle(v_1, v_2)$$

**b :** For n=2 and k=2,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} + \triangle(v_1^*, v_2^*)_{\varepsilon^1}\varepsilon$$

where $\triangle(v_1^*, v_2^*)_{\varepsilon^1} = \triangle(v_1, v_{2,\varepsilon}) + \triangle(v_{1,\varepsilon}, v_2)$

**c :** For n=2 and k=3,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} + \triangle(v_1^*, v_2^*)_{\varepsilon^1}\varepsilon + \triangle(v_1^*, v_2^*)_{\varepsilon^2}\varepsilon^2$$

where $\triangle(v_1^*, v_2^*)_{\varepsilon^2} = \triangle(v_1, v_{2,\varepsilon^2}) + \triangle(v_{1,\varepsilon}, v_{2,\varepsilon}) + \triangle(v_{1,\varepsilon^2}, v_2)$.
Following is cross ratios in $F[\varepsilon]_k$ [11]

$$\mathbf{r}(v_0^*, v_1^*, v_2^*, v_3^*) = (r_{\varepsilon^0} + r_{\varepsilon^1}\varepsilon + \ldots + r_{\varepsilon^{k-1}}\varepsilon^{k-1})(v_0^*, v_1^*, v_2^*, v_3^*)$$

Where

$$r_{\varepsilon^0}(v_0^*, v_1^*, v_2^*, v_3^*) = r(v_0, v_1, v_2, v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}, \tag{2.5}$$

$$r_{\varepsilon^1}(v_0^*, v_1^*, v_2^*, v_3^*) = \frac{\{\triangle(v_0^*, v_3^*)\triangle(v_1^*, v_2^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} - r(v_0, v_1, v_2, v_3)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} \tag{2.6}$$

$$\begin{aligned} r_{\varepsilon^2}(v_0^*, v_1^*, v_2^*, v_3^*) = {} & \frac{\{\triangle(v_0^*, v_3^*)\triangle(v_1^*, v_2^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} - r(v_0^*, v_1^*, v_2^*, v_3^*)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} \\ & - r(v_0, v_1, v_2, v_3)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} \end{aligned} \tag{2.7}$$

and so on

2.4. **Classical Polylog Chain Complexes.** The p-logarithm function is a series defined as $Li_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}, \quad z \leq 1$, with property $\log(a) + \log(b) - \log(ab) = 0$. Assume that $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is a free abelian group generated by a symbol [x] where symbol $[x]$ means logarithms of $x$ [5].

**Definition 2.5.** *$\mathcal{B}(F)$ is a Scissor congruence group defined as $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is quotient by expression called Abel five terms relation* $[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-y^{-1}}{1-x^{-1}}\right]+\left[\frac{1-y}{1-x}\right], x \neq y$ *and* $x, y \neq 0, 1$

2.5.1. *Bloch Group for Weight-1.* Bloch [1] defined $\mathsf{R}_1(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$, generated by the relation $\{xy\} - \{x\} - \{y\}$, $(x, y \in F^\times)$. Then $\mathcal{B}_1(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is called Bloch group for weight 1 defined as $\mathcal{B}_1(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\Big\langle \mathsf{R}_1(F)\Big\rangle$. He defined a map $\delta_1 : \mathcal{B}_1(F) \to F^\times$, defined as $\delta_1 : [x] \to x$. This map is also an isomorphism. So, $\mathcal{B}_1(F) \cong F^\times$.

2.5.2. *Bloch-Suslin Chain Complex.* Let $\mathsf{R}_2(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is a sub group generated by the relation $\sum_{i=0}^{4}(-1)^i r(v_0, ..., \hat{v_i}, ..., v_4)$, define a morphism $\delta_2 : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \wedge^2 F^\times$, where $\delta_2 : [x] \to (1-x) \wedge x$. This helped to defined a dilogarithm Bloch group for weight 2 as $\mathcal{B}_2(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\langle \mathsf{R}_2(F)\rangle$ which connected $\mathcal{B}_2(F)$ with $\wedge^2 F^\times$ to form a chain complex called the Bloch-Suslin complex [1, 10].

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times$$

where $\delta$ is an induced map defined as

$$\delta : [v]_2 \to (1-v) \wedge v$$

2.5.3. *Goncharov Chain Complex for Weight-3.* Goncharov [6,7] defined $\mathsf{R}_3(F) \subset Z[\mathbf{P}^1_F/\{0, 1,\infty\}]$, such that

$$\mathsf{R}_3(F) = \sum_{i=0}^{6}(-1)^i Alt_6\Big[\frac{(v_0,v_1,v_3)(v_1,v_2,v_4)(v_0,v_2,v_5)}{(v_0,v_1,v_4)(v_1,v_2,v_5)(v_0,v_2,v_3)}\Big] \tag{2.8}$$

For weight 3 Goncharov [7–9] introduced a group $\mathcal{B}_3(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\langle \mathsf{R}_3(F)\rangle$. Chain complex for weight 3 is given by

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times$$

where

$$\delta : [v]_3 \to [v]_2 \otimes v$$

**Lemma 2.6.** $\delta \circ \delta = 0$ (see [7])

2.6.1. *Weight-n.* Goncharov [7] generalized $\mathcal{B}(F)$ as $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/ < \mathsf{R}_n(F) >$, where $\mathsf{R}_n(F) \subset Z[\mathbf{P}^1_F]$ is the kernel of the map $\delta_n : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \mathcal{B}_{n-1}(F) \otimes F^\times$. Then Goncharov generalized Bloch-Suslin complex for any weight $n$, called Goncharov complex

$$\mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_{n-1}} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta_{n-2}} \dots \xrightarrow{\delta_2} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta_1} \wedge^n(F^\times) \tag{2.9}$$

**Lemma 2.7.** $\delta_{n-1} \circ \delta_n = 0$ (see [7, 17])

2.8. **Tangent Groups and Generalized Tangent Complexes.**

2.8.1. *First Order Tangent Group.* For any elements $x, x' \in F$ and $\left\langle x; x' \right]_2 = [x + x'\varepsilon] - [x] \in \mathbb{Z}[F[\varepsilon]_2$, Cathelineau [4] introduced $T\mathcal{B}_2(F)$ as a first order tangent group. It is a Z-module generated by the elements $\left\langle x; x' \right]_2 \in \mathbb{Z}[F[\varepsilon]_2$ an quotient by the five term relation

$$\left\langle x; x' \right] - \left\langle y; y' \right] + \left\langle \frac{y}{x}; \left(\frac{y}{x}\right)' \right] - \left\langle \frac{1-y}{1-x}; \left(\frac{1-y}{1-x}\right)' \right] + \left\langle \frac{x(1-y)}{y(1-x)}; \left(\frac{x(1-y)}{y(1-x)}\right)' \right]$$

where $x, y \neq 0, 1$ and $x \neq y$ [4, 11].

2.8.2. *Tangent Complex to Bloch-Suslin Chain Complex for Weight 2.* Cathelineau [4] introduced following Tangent complex to Bloch-Suslin complex

$$T\mathcal{B}_2(F) \xrightarrow{\delta_\varepsilon} F \otimes F^\times \oplus \wedge^2 F^\times$$

where

$$\delta_\varepsilon : \left\langle x, y \right]_2 = \left(\frac{y}{x} \otimes (1-x) + \frac{y}{(1-x)} \otimes x\right) + \left(\frac{y}{(1-x)} \wedge \frac{y}{x}\right)$$

see [4, 20].

2.8.3. *Tangent Complex to Goncharov Chain Complex for Weight 3.* Cathelineau [4] introduced following Tangent complex to Goncharov chain complex for weight 3

$$T\mathcal{B}_3(F) \xrightarrow{\delta_\varepsilon} (T\mathcal{B}_2(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) \xrightarrow{\delta_\varepsilon} F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times$$

2.8.4. *Generalized Tangent Complex for any Tangent Group* $\mathbf{T\mathcal{B}_n(F)}$. Cathelineau [4] generalized following Tangent complex to Goncharov chain complex for any weight $n$

$$T\mathcal{B}_n(F) \xrightarrow{\delta_{n,\varepsilon}} \begin{matrix} T\mathcal{B}_{n-1}(F) \otimes F^\times \\ \oplus \\ F \otimes \mathcal{B}_{n-1}(F) \end{matrix} \xrightarrow{\delta_{(n-1),\varepsilon}} \cdots \xrightarrow{\delta_{1,\varepsilon}} \begin{matrix} T\mathcal{B}_2(F) \otimes \wedge^{n-2} F^\times \\ \oplus \\ F \otimes \mathcal{B}_2(F) \otimes \wedge^{n-3} F^\times \end{matrix} \xrightarrow{\delta_\varepsilon} (F \otimes \wedge^{n-1} F^\times) \oplus (\wedge^n F)$$

2.9. **Geometry of Tangent and Configuration Chain Complexes up to Weight 3.**

2.9.1. *Geometry for Weight 2.* As defined in [20], the geometry of Grassmannian configuration and Goncharov motivic in weight-2 is represented as

$$\begin{array}{ccccc} G_5(3) & \xrightarrow{p} & G_4(2) & \xrightarrow{g^2_{1,\varepsilon}} & T\mathcal{B}_2(F) \\ \downarrow d & & \downarrow d & & \downarrow \delta_\varepsilon \\ G_4(3) & \xrightarrow{p} & G_3(2) & \xrightarrow{g^2_{0,\varepsilon}} & F\otimes F^\times \oplus \wedge^2 F^\times \end{array} \tag{C}$$

**Lemma 2.10.** *The diagram* B *is bi-complex and commutative* [20].

2.10.1. *Geometry for Weight 3.* The geometry of Grassmannian and Goncharov motivic for weight-3 is presented in [20] as follows:

$$\begin{array}{ccccc} G_7(3) & \xrightarrow{d} & G_6(3) & & \\ \downarrow p & & \downarrow p & & \\ G_6(2) & \xrightarrow{d} & G_5(2) & \xrightarrow{g^3_{1,\varepsilon}} & T\mathcal{B}_2(F)\otimes F^\times \oplus F\otimes \mathcal{B}_2(F) \\ \downarrow p & & \downarrow p & & \downarrow \delta_\varepsilon \\ G_5(1) & \xrightarrow{d} & G_4(1) & \xrightarrow{g^3_{0,\varepsilon}} & F\otimes \wedge^2 F^\times \oplus \wedge^3 F^\times \end{array} \tag{D}$$

**Lemma 2.11.** *The diagram* C *is bi-complex and commutative* [20].

## 3. Geometry of Tangent Groups and Configuration Chain Complexes for Weight 4 & 5

### 3.1. **Geometry for Weight 4.** Geometry for weight 4 is defined as follows

$$\begin{array}{ccccc} G_7(\mathcal{A}^5_{F[\varepsilon]_r}) & \xrightarrow{p} & G_6(\mathcal{A}^4_{F[\varepsilon]_2}) & \xrightarrow{g^4_{1,\varepsilon}} & T\mathcal{B}_2(F)\otimes \wedge^2 F^\times \oplus F\otimes \mathcal{B}_2(F)\otimes F^\times \\ \downarrow d & & \downarrow d & & \downarrow \delta_\varepsilon \\ G_6(\mathcal{A}^5_{F[\varepsilon]_r}) & \xrightarrow{p} & G_5(\mathcal{A}^4_{F[\varepsilon]_2}) & \xrightarrow{g^4_{0,\varepsilon}} & F\otimes \wedge^3 F^\times \oplus \wedge^4 F^\times \end{array} \tag{E}$$

where, $g^4_{0\varepsilon}(v^*_0,...,v^*_4) = g^4_{0_1}(v^*_0,...,v^*_4) + g^4_{0_2}(v^*_0,...,v^*_4)$

$$g^4_{0_1}(v^*_0,...,v^*_4) = \sum_{i=j+1}^{4} (-1)^{i+1} \frac{\triangle(v^*_0,...,\hat{v^*}_i,...,v^*_4)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)} \ (i \ mod \ 5), \tag{3.10}$$

$$g^4_{0_2}(v^*_0,...,v^*_4) = \sum_{j=0}^{4} (-1)^{j+1} \bigwedge_{\substack{j\neq i \\ j=0}}^{4} \frac{\triangle(v^*_0,...,\hat{v^*}_j,...,v^*_4)\varepsilon}{\triangle(v_0,...,\hat{v}_j,...,v_4)} \ (i \ mod \ 5) \tag{3.11}$$

and

$$g^4_{1\varepsilon}(v^*_0,...,v^*_5) = -\frac{1}{10}\sum_{i\neq j}^{5}(-1)^i\Big(\Big\langle r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5); r_\varepsilon(v^*_i,v^*_j|v^*_0,$$
$$...,\hat{v^*}_i,\hat{v^*}_j,...,v^*_5)\Big]_2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\wedge$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{i\neq r\\ i=0}}^{5}\frac{\triangle(v^*_0,...,\hat{v^*}_i,\hat{v^*}_r,...,v^*_5)_\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{j\neq r\\ j=0}}^{5}\frac{\triangle(v^*_0,...,\hat{v^*}_j,\hat{v^*}_r,...,v^*_5)_\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\Big)\ (mod\ 6). \tag{3.12}$$

**Lemma 3.2.** *The right square of diagram* E *is commutative.*

$$\begin{array}{ccc} G_6(\mathcal{A}^4_{F[\varepsilon]_2}) & \xrightarrow{\quad d \quad} & G_5(\mathcal{A}^4_{F[\varepsilon]_2}) \\ \downarrow{\scriptstyle g^4_{1,\varepsilon}} & & \downarrow{\scriptstyle g^4_{0.\varepsilon}} \\ T\mathcal{B}_2(F)\otimes F^\times \oplus F\otimes\mathcal{B}_2(F)\otimes F^\times & \xrightarrow{\ \delta_\varepsilon\ } & F\otimes\wedge^3F^\times\oplus\wedge^4F^\times \end{array}$$

*Proof.* Let us assume $(v^*_0,...,v^*_5)\in G_6(\mathcal{A}^4_{F[\varepsilon]_r})$ and apply morphism $d$

$$\begin{aligned} d(v^*_0,...,v^*_5) &= \sum_{i=0}^{5}(-1)^j(v^*_0,...,\hat{v^*}_i,...,v^*_5) \\ &= (v^*_1,v^*_2,v^*_3,v^*_4,v^*_5) - (v^*_0,v^*_2,v^*_3,v^*_4,v^*_5) + (v^*_0,v^*_1,v^*_3,v^*_4,v^*_5) \\ &\quad - (v^*_0,v^*_1,v^*_2,v^*_4,v^*_5) + (v^*_0,v^*_1,v^*_2,v^*_3,v^*_5) - (v^*_0,v^*_1,v^*_2,v^*_3,v^*_4) \end{aligned} \tag{3.13}$$

now first apply map $g^4_{01}$, we get,

$$\begin{aligned} g^4_{01}\circ d(v^*_0,...,v^*_5) =& g^4_{01}(v^*_1,v^*_2,v^*_3,v^*_4,v^*_5) - g^4_{01}(v^*_0,v^*_2,v^*_3,v^*_4,v^*_5) + \\ & g^4_{01}(v^*_0,v^*_1,v^*_3,v^*_4,v^*_5) - g^4_{01}(v^*_0,v^*_1,v^*_2,v^*_4,v^*_5) + \\ & g^4_{01}(v^*_0,v^*_1,v^*_2,v^*_3,v^*_5) - g^4_{01}(v^*_0,v^*_1,v^*_2,v^*_3,v^*_4) \end{aligned} \tag{3.14}$$

Expand by applying map $g^4_{01}$

$$g^4_{01}\circ d(v^*_0,...,v^*_5) = \sum_{\substack{i\neq 0\\ i=1}}^{5}(-1)^{i+1}\frac{\triangle(v^*_1,...,\hat{v^*}_i,...,v^*_5)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 1\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 2\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 3\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 4\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}. \tag{3.15}$$

now compose map $g_{02}^4$ with $d(v_0^*,...,v_5^*)$, we get,

$$\begin{aligned} g_{02}^4\circ d(v_0^*,...,v_5^*) =& g_{02}^4(v_1^*,v_2^*,v_3^*,v_4^*,v_5^*)-g_{02}^4(v_0^*,v_2^*,v_3^*,v_4^*,v_5^*)+\\ & g_{02}^4(v_0^*,v_1^*,v_3^*,v_4^*,v_5^*)-g_{02}^4(v_0^*,v_1^*,v_2^*,v_4^*,v_5^*)+\\ & g_{02}^4(v_0^*,v_1^*,v_2^*,v_3^*,v_5^*)-g_{02}^4(v_0^*,v_1^*,v_2^*,v_3^*,v_4^*) \end{aligned} \tag{3.16}$$

Expand by applying map $g_{02}^4$

$$g_{02}^4\circ d(v_0^*,...,v_5^*)=\sum_{\substack{i\neq 0\\ i=1}}^{5}(-1)^{i+1}\frac{\triangle(v_1^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 2\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 3\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 4\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{4}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)}. \tag{3.17}$$

Combine Eq.(3.15) and Eq.(3.17), then

$$g_{0\varepsilon}^4\circ d(v_0,...,5_4)=\sum_{\substack{i\neq 0\\ i=1}}^{5}(-1)^{i+1}\frac{\triangle(v_1^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$
$$\sum_{\substack{i\neq 1\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{4}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)}+$$

$$\sum_{\substack{i\neq 0\\ i=1}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{4}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)}. \quad (3.18)$$

Take $(v_0^*,...,v_5^*)\in G_6(\mathcal{A}^4_{F[\varepsilon]_r})$ again and apply morphism $g^4_{1\varepsilon}$

$$g^4_{1\varepsilon}(v_0^*,...,v_5^*) = -\frac{1}{10}\sum_{i\neq j}^{5}(-1)^i\Big(\Big\langle r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5); r_\varepsilon(v_i^*,v_j^*|v_0^*,$$

$$...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\Big]_2\otimes$$

$$\prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\wedge\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)+$$

$$\sum_{\substack{i\neq r\\ i=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,...,v_5^*)_\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)}\otimes[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes$$

$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)+\sum_{\substack{j\neq r\\ j=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,...,v_5^*)_\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)}\otimes$$

$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes\prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\Big) \quad (3.19)$$

now apply morphism $\delta_{varepsilon}$, then

$$\delta_\varepsilon\circ g^4_{1\varepsilon}(v_0^*,...,v_5^*) = -\frac{1}{10}\sum_{i\neq j}^{5}(-1)^i\Big(\Big\langle r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5); r_\varepsilon(v_i^*,v_j^*|v_0^*,$$

$$...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\Big]_2\otimes$$

$$\prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\wedge\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)+$$

$$\sum_{\substack{i\neq r\\ i=0}}^{5} \frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,...,v_5^*)_\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)} \otimes [r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2 \otimes$$
$$\prod_{j\neq r}^{5} \triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{j\neq r\\ j=0}}^{5} \frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,...,v_5^*)_\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)} \otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2 \otimes \prod_{i\neq r}^{5} \triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\Big). \tag{3.20}$$

Using wedge and Siegel cross ratio properties [21], then above Eq.(3.20) becomes

$$\delta_\varepsilon \circ g^4_{1\varepsilon}(v_0^*,...,v_5^*) = \sum_{\substack{i\neq 0\\ i=1}}^{5} (-1)^{i+1} \frac{\triangle(v_1^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)} \wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)} +$$
$$\sum_{\substack{i\neq 1\\ i=0}}^{5} (-1)^{i+1} \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} +$$
$$\sum_{\substack{i\neq 2\\ i=0}}^{5} (-1)^{i+1} \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)} \wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)} +$$
$$\sum_{\substack{i\neq 3\\ i=0}}^{5} (-1)^{i+1} \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} +$$
$$\sum_{\substack{i\neq 4\\ i=0}}^{5} (-1)^{i+1} \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} +$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{4} (-1)^{i+1} \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)}+$$

$$\sum_{\substack{i\neq 0\\ i=1}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{4}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)}. \tag{3.21}$$

So from Eq.(3.15) and Eq.(3.21), it is proved that the above diagram is commutative. □

3.3. **Geometry for Weight 5.** For weight 5 we have following commutative diagram

$$\begin{array}{ccccc} G_8(\mathcal{A}^6_{F[\varepsilon]_2}) & \xrightarrow{p} & G_7(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{1,\varepsilon}} & T\mathcal{B}_2(F)\otimes\wedge^3F^\times\oplus F\otimes\mathcal{B}_2(F)\otimes\wedge^2F^\times \\ \downarrow d & & \downarrow d & & \downarrow \delta_\varepsilon \\ G_7(\mathcal{A}^6_{F[\varepsilon]_2}) & \xrightarrow{p} & G_6(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{0,\varepsilon}} & F\otimes\wedge^4F^\times\oplus\wedge^5F^\times \end{array} \quad \text{(F)}$$

where, $g^5_{0\varepsilon}(v^*_0,...,v^*_5)=g^5_{0_1}(v^*_0,...,v^*_5)+g^5_{0_2}(v^*_0,...,v^*_5)$

$$g^5_{0_1}(v^*_0,...,v^*_5)=\sum_{i=j+1}^{5}(-1)^{i+1}\frac{\triangle(v^*_0,...,\hat{v^*}_i,...,v^*_5)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}\wedge \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}(i\ mod\ 6), \tag{3.22}$$

$$g^5_{0_2}(v^*_0,...,v^*_5)=\sum_{j=0}^{5}(-1)^{j+1}\bigwedge_{\substack{j\neq i\\ j=0}}^{5}\frac{\triangle(v^*_0,...,\hat{v^*}_j,...,v^*_5)\varepsilon}{\triangle(v_0,...,\hat{v}_j,...,v_5)}\ (i\ mod\ 6) \tag{3.23}$$

and

$$g^5_{1\varepsilon}(v^*_0,...,v^*_6)=\frac{1}{15}\sum_{i\neq j}^{6}(-1)^i\Big(\Big\langle r(v_i,v_j,v_j|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6);r_\varepsilon(v^*_i,v^*_j,v^*_k|v^*_0,..., \hat{v^*}_i,\hat{v^*}_j,\hat{v^*}_k,...,v^*_6)\Big]_2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge \prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6) + \sum_{\substack{i\neq r\neq s\\ i=0}}^{6}\frac{\triangle(v^*_0,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v^*_6)\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}\otimes[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k, ...,v_6)]_2\otimes\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s, ...,v_6)+\sum_{\substack{j\neq r\neq s\\ j=0}}^{6}\frac{\triangle(v^*_0,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v^*_6)\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}\otimes[r(v_i,v_j,v_k|v_0,...,\hat{v}_i, \hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,$$

$$\hat{v}_r, \hat{v}_s, ..., v_6) + \sum_{\substack{k\neq r\neq s\\k=0}}^{6} \frac{\triangle(v_0^*, ..., \hat{v^*}_k, \hat{v^*}_r, \hat{v^*}_s, ..., v_6^*)\varepsilon}{\triangle(v_0, ..., \hat{v}_k, \hat{v}_r, \hat{v}_s, ..., v_6)} \otimes [r(v_i, v_j, v_k|v_0,$$
$$..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{i\neq r\neq s}^{5} \triangle(v_0, ..., \hat{v}_i, \hat{v}_r, \hat{v}_s, ..., v_6) \wedge \prod_{j\neq r\neq s}^{6} \triangle(v_0,$$
$$..., \hat{v}_j, \hat{v}_r, \hat{v}_s, ..., v_6)\ (mod\ 6). \tag{3.24}$$

**Lemma 3.4.** *The right square of diagram F is commutative.*

$$\begin{array}{ccc} G_7(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{1,\varepsilon}} & T\mathcal{B}_2(F) \otimes \wedge^3 F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes \wedge^2 F^\times \\ \downarrow d & & \downarrow \delta_\varepsilon \\ G_6(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{0,\varepsilon}} & F \otimes \wedge^4 F^\times \oplus \wedge^5 F^\times \end{array}$$

*Proof.* Let us assume $(v_0^*, ..., v_6^*) \in G_7(\mathcal{A}^5_{F[\varepsilon]_r})$ and apply morphism $d$

$$d(v_0^*, ..., v_6^*) = \sum_{i=0}^{6} (-1)^j (v_0^*, ..., \hat{v^*}_i, ..., v_6^*)$$

$\Longrightarrow$

$$\begin{aligned} d(v_0^*, ..., v_6^*) =& (v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*) - (v_0^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*) + \\ & (v_0^*, v_1^*, v_3^*, v_4^*, v_5^*, v_6^*) - (v_0^*, v_1^*, v_2^*, v_4^*, v_5^*, v_6^*) + \\ & (v_0^*, v_1^*, v_2^*, v_3^*, v_5^*, v_6^*) - (v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_6^*) + \\ & (v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_5^*) \end{aligned} \tag{3.25}$$

now first apply map $g_{01}^5$, we get,

$$\begin{aligned} g_{01}^5 \circ d(v_0^*, ..., v_6^*) =& g_{01}^5(v_1^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*) - g_{01}^5(v_0^*, v_2^*, v_3^*, v_4^*, v_5^*, v_6^*) + \\ & g_{01}^5(v_0^*, v_1^*, v_3^*, v_4^*, v_5^*, v_6^*) - g_{01}^5(v_0^*, v_1^*, v_2^*, v_4^*, v_5^*, v_6^*) + \\ & g_{01}^5(v_0^*, v_1^*, v_2^*, v_3^*, v_5^*, v_6^*) - g_{01}^5(v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_6^*) + \\ & g_{01}^5(v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_5^*) \end{aligned} \tag{3.26}$$

Expand by applying map $g_{01}^5$

$$\begin{aligned} g_{01}^5 \circ d(v_0^*, ..., v_6^*) = \sum_{\substack{i\neq 0\\i=1}}^{6} (-1)^{i+1} & \frac{\triangle(v_1^*, ..., \hat{v^*}_i, ..., v_6^*)\varepsilon}{\triangle(v_1, ..., \hat{v}_i, ..., v_6)} \otimes \frac{\triangle(v_1, ..., \hat{v}_{i+1}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+2}, ..., v_6)} \wedge \\ & \frac{\triangle(v_1, ..., \hat{v}_{i+2}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+3}, ..., v_6)} \wedge \frac{\triangle(v_1, ..., \hat{v}_{i+3}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+4}, ..., v_6)} \wedge \\ & \frac{\triangle(v_1, ..., \hat{v}_{i+4}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+5}, ..., v_6)} + \end{aligned}$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 6\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}. \tag{3.27}$$

now compose map $g^5_{02}$ with $d(v^*_0,...,v^*_6)$, we get,

$$\begin{aligned} g^5_{02}\circ d(v^*_0,...,v^*_6) =& g^5_{02}(v^*_1,v^*_2,v^*_3,v^*_4,v^*_5,v^*_6) - g^5_{02}(v^*_0,v^*_2,v^*_3,v^*_4,v^*_5,v^*_6)+ \\ & g^5_{02}(v^*_0,v^*_1,v^*_3,v^*_4,v^*_5,v^*_6) - g^5_{02}(v^*_0,v^*_1,v^*_2,v^*_4,v^*_5,v^*_6)+ \\ & g^5_{02}(v^*_0,v^*_1,v^*_2,v^*_3,v^*_5,v^*_6) - g^5_{02}(v^*_0,v^*_1,v^*_2,v^*_3,v^*_4,v^*_5). \end{aligned} \tag{3.28}$$

Expand by applying map $g^5_{02}$

$$\begin{aligned} g^5_{02}\circ d(v^*_0,...,v^*_6) =& \sum_{\substack{i\neq 0\\ i=1}}^{6}(-1)^{i+1}\frac{\triangle(v^*_1,...,\hat{v^*}_i,...,v^*_6)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_6)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)} \wedge \\ & \frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)} \wedge \\ & \frac{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+5},...,v_6)} + \\ & \sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v^*_0,...,\hat{v^*}_i,...,v^*_6)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} + \\ & \sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v^*_0,...,\hat{v^*}_i,...,v^*_6)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} + \\ & \sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v^*_0,...,\hat{v^*}_i,...,v^*_6)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge \\ & \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} + \end{aligned}$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 6\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}. \tag{3.29}$$

Add Eq.(3.27) and Eq.(3.29), then,

$$g_{0,\varepsilon}^5\circ d(v_0^*,...,v_6^*)=\sum_{\substack{i\neq 0\\ i=1}}^{6}(-1)^{i+1}\frac{\triangle(v_1^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \otimes \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)} +$$

$$\sum_{\substack{i\neq 0\\ i=1}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_6)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)} \wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)} \wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$
$$\sum_{\substack{i\neq 5\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}. \tag{3.30}$$

Take $(v_0^*,...,v_6^*) \in G_7(\mathcal{A}^5_{F[\varepsilon]_r})$ again and compose with morphism $g^5_{1,\varepsilon}$, then

$$\begin{aligned} g^5_{1,\varepsilon}(v_0^*,...,v_6^*) = \frac{1}{15}\sum_{i\neq j}^{6}(-1)^i\Big(\Big\langle r(v_i,v_j,v_j|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6); r_\varepsilon(v_i^*,v_j^*,v_k^*|v_0^*, \\ ...,\hat{v^*}_i,\hat{v^*}_j,\hat{v^*}_k,...,v_6^*)\Big]_2 \otimes \prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge \\ \prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)+ \\ \sum_{\substack{i\neq r\neq s\\ i=0}}^{6}\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}\otimes[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,..., \\ v_6)]_2\otimes\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,..., \\ v_6)+\sum_{\substack{j\neq r\neq s\\ j=0}}^{6}\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}\otimes[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j, \\ \hat{v}_k,...,v_6)]_2\otimes\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r, \\ \hat{v}_s,...,v_6)+\sum_{\substack{k\neq r\neq s\\ k=0}}^{6}\frac{\triangle(v_0^*,...,\hat{v^*}_k,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)}\otimes[r(v_i,v_j,v_k|v_0,..., \\ \hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{j\neq r\neq s}^{6}\triangle(v_0,..., \\ \hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6) \end{aligned} \tag{3.31}$$

now apply morphism $\delta_\varepsilon$, then

$$\begin{aligned} \delta_\varepsilon\circ g^5_{1,\varepsilon}(v_0^*,...,v_6^*) = \frac{1}{15}\sum_{i\neq j}^{6}(-1)^i\Big(\Big\langle r(v_i,v_j,v_j|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6); r_\varepsilon(v_i^*,v_j^*, \\ v_k^*|v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,\hat{v^*}_k,...,v_6^*)\Big]_2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,..., \end{aligned}$$

$$v_6) \wedge \prod_{j\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_j, \hat{v}_r, \hat{v}_s, ..., v_6) \wedge \prod_{k\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_k, \hat{v}_r,$$

$$\hat{v}_s, ..., v_6) + \sum_{\substack{i\neq r\neq s\\ i=0}}^{6} \frac{\triangle(v_0^*, ..., \hat{v^*}_i, \hat{v^*}_r, \hat{v^*}_s, ..., v_6^*)\varepsilon}{\triangle(v_0, ..., \hat{v}_i, \hat{v}_r, \hat{v}_s, ..., v_6)} \otimes [r(v_i, v_j,$$

$$v_k|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{j\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_j, \hat{v}_r, \hat{v}_s, ..., v_6)\wedge$$

$$\prod_{k\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_k, \hat{v}_r, \hat{v}_s, ..., v_6)+$$

$$\sum_{\substack{j\neq r\neq s\\ j=0}}^{6} \frac{\triangle(v_0^*, ..., \hat{v^*}_j, \hat{v^*}_r, \hat{v^*}_s, ..., v_6^*)\varepsilon}{\triangle(v_0, ..., \hat{v}_j, \hat{v}_r, \hat{v}_s, ..., v_6)} \otimes [r(v_i, v_j, v_k|v_0, ..., \hat{v}_i,$$

$$\hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{k\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_k, \hat{v}_r, \hat{v}_s, ..., v_6) \wedge \prod_{i\neq r\neq s}^{6} \triangle(v_0,$$

$$..., \hat{v}_i, \hat{v}_r, \hat{v}_s, ..., v_6) + \sum_{\substack{k\neq r\neq s\\ k=0}}^{6} \frac{\triangle(v_0^*, ..., \hat{v^*}_k, \hat{v^*}_r, \hat{v^*}_s, ..., v_6^*)\varepsilon}{\triangle(v_0, ..., \hat{v}_k, \hat{v}_r, \hat{v}_s, ..., v_6)}\otimes$$

$$[r(v_i, v_j, v_k|v_0, ..., \hat{v}_i, \hat{v}_j, \hat{v}_k, ..., v_6)]_2 \otimes \prod_{i\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_i, \hat{v}_r, \hat{v}_s,$$

$$..., v_6) \wedge \prod_{j\neq r\neq s}^{6} \triangle(v_0, ..., \hat{v}_j, \hat{v}_r, \hat{v}_s, ..., v_6) \tag{3.32}$$

Using wedge and Siegel cross ratio properties [21], then above Eq.(3.32) becomes

$$\delta_\varepsilon \circ g^5_{1,\varepsilon}(v_0^*, ..., v_6^*) = \sum_{\substack{i\neq 0\\ i=1}}^{6}(-1)^{i+1} \frac{\triangle(v_1^*, ..., \hat{v^*}_i, ..., v_6^*)\varepsilon}{\triangle(v_1, ..., \hat{v}_i, ..., v_6)} \otimes \frac{\triangle(v_1, ..., \hat{v}_{i+1}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+2}, ..., v_6)}\wedge$$

$$\frac{\triangle(v_1, ..., \hat{v}_{i+2}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+3}, ..., v_6)} \wedge \frac{\triangle(v_1, ..., \hat{v}_{i+3}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+4}, ..., v_6)}\wedge$$

$$\frac{\triangle(v_1, ..., \hat{v}_{i+4}, ..., v_6)}{\triangle(v_1, ..., \hat{v}_{i+5}, ..., v_6)}+$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1} \frac{\triangle(v_0^*, ..., \hat{v^*}_i, ..., v_6^*)\varepsilon}{\triangle(v_0, ..., \hat{v}_i, ..., v_6)} \otimes \frac{\triangle(v_0, ..., \hat{v}_{i+1}, ..., v_6)}{\triangle(v_0, ..., \hat{v}_{i+2}, ..., v_6)}\wedge$$

$$\frac{\triangle(v_0, ..., \hat{v}_{i+2}, ..., v_6)}{\triangle(v_0, ..., \hat{v}_{i+3}, ..., v_6)} \wedge \frac{\triangle(v_0, ..., \hat{v}_{i+3}, ..., v_6)}{\triangle(v_0, ..., \hat{v}_{i+4}, ..., v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\otimes\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}+$$

$$\sum_{\substack{i\neq 0\\ i=1}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_1,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_1,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_1,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 1\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 2\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 3\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 4\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)}+$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{6}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}\wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_6)} +$$

$$\sum_{\substack{i\neq 5\\ i=0}}^{5}(-1)^{i+1}\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge$$

$$\frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}. \quad (3.33)$$

□

From Eq.(3.30) and Eq.(3.33), it can be seen that the map of morphism between the tangential and configuration chain complex for weight 5 is commutative.

## 4. CONCLUSION

In this research work, new morphisms have been presented for 4 and 5 dimensional affine space to define the geometry between tangential and configuration chain complexes. The composite maps for weight 4 and 5 are found to be commutative. In a similar technique, the tangential group $T\mathcal{B}_n(F)$ for any weight "n" can be defined by relating them with suitable complex.

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